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Note on a weakly over-penalised symmetric interior penalty method on anisotropic meshes for the Poisson equation, Ver. 1

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Abstract The purpose is to make an easy-to-understand note of "Special Topics in Finite Element Methods." There might be typos and mistakes. Therefore, I do not take any responsibility for unauthorised use.

Keywords Poisson equation · WOPSIP method · CR finite element method · RT finite element method · Anisotropic meshes

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1 Preliminaries

Throughout, we denote by c a constant independent of h (defined later) and of the angles and aspect ratios of simplices unless specified otherwise, and all constants c are bounded if the maximum angle is bounded. These values may change in each context. The notation $\mathbb{R}_+$ denotes the set of positive real numbers.

1.1 Continuous problems

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. The Poisson problem is to find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

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where $f \in L^2(\Omega)$ is a given function. The variational formulation for the Poisson equations (1.1) is as follows. Find $u \in H_0^1(\Omega)$ such that

$$a(u,\varphi) := \int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in H_0^1(\Omega). \tag{1.2}$$

By the Lax–Milgram lemma, there exists a unique solution $u \in H_0^1(\Omega)$ for any $f \in L^2(\Omega)$ and it holds that

$$|u|_{H^1(\Omega)} \leq C_P(\Omega)\|f\|,$$

where $C_P(\Omega)$ is the Poincaré constant depending on Ω. Furthermore, if Ω is convex, then $u \in H^2(\Omega)$ and

$$|u|_{H^2(\Omega)} \leq \|\Delta u\|. \tag{1.3}$$

The proof can be found in, for example, [29, Theorem 3.1.1.2, Theorem 3.2.1.2].

1.2 Meshes, mesh faces, averages and jumps

Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices such as $\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T$ with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \mathrm{diam}(T)$. For simplicity, we assume that $\mathbb{T}_h$ is conformal: that is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes.

Let $\mathcal{F}_h^i$ be the set of interior faces and $\mathcal{F}_h^\partial$ the set of the faces on the boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows. (i) If $F \in \mathcal{F}_h^i$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$, let n_1 and n_2 be the outward unit normals of T_1 and T_2, respectively. Then, n_F is either of $\{n_1, n_2\}$; (ii) If $F \in \mathcal{F}_h^\partial$, n_F is the unit outward normal n to $\partial\Omega$. For a simplex $T \subset \mathbb{R}^d$, let $\mathcal{F}_T$ be the collection of the faces of T.

Here, we consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. We define a broken (piecewise) Hilbert space as

$$H^m(\mathbb{T}_h)^q := \{v \in L^2(\Omega)^q : \ v|_T \in H^1(T)^q \quad \forall T \in \mathbb{T}_h\}, \quad m \in \mathbb{N}$$

with a norm

$$|v|_{H^m(\mathbb{T}_h)^q} := \left(\sum_{T \in \mathbb{T}_h} |v|^2_{H^m(T)^q} \right)^{\frac{1}{2}}.$$

When $q = 1$, we denote $H^m(\mathbb{T}_h) := H^m(\mathbb{T}_h)^1$. Let $\varphi \in H^1(\mathbb{T}_h)$. Suppose that $F \in \mathcal{F}_h^i$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$. Set $\varphi_1 := \varphi|_{T_1}$ and $\varphi_2 := \varphi|_{T_2}$. Set two nonnegative real numbers $\omega_{T_1,F}$ and $\omega_{T_2,F}$ such that

$$\omega_{T_1,F} + \omega_{T_2,F} = 1.$$

The jump and the skew-weighted average of φ across F is then defined as

$$[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_1 - \varphi_2, \quad \{\{\varphi\}\}_{\overline{\omega}} := \{\{\varphi\}\}_{\overline{\omega},F} := \omega_{T_2,F}\varphi_1 + \omega_{T_1,F}\varphi_2.$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial \Omega$, $[\![\varphi]\!]_F := \varphi|_T$ and $\{\{\varphi\}\}_{\overline{\omega}} := \varphi|_T$. For any $v \in H^1(\mathbb{T}_h)^d$, we use the notation

$$[\![v \cdot n]\!] := [\![v \cdot n]\!]_F := v_1 \cdot n_1 + v_2 \cdot n_2, \ \{\{v\}\}_\omega := \{\{v\}\}_{\omega,F} := \omega_{T_1,F} v_1 + \omega_{T_2,F} v_2,$$
$$[\![v]\!] := [\![v]\!]_F := v_1 - v_2,$$

for the jump of the normal component of v, the weighted average of v, and the junp of v. For $F \in \mathcal{F}_h$, $[\![v \cdot n]\!] := v \cdot n$, $\{\{v\}\}_\omega := v$ and $[\![v]\!] :=$, where n is the outward normal.

We define a broken gradient operator as follows. For $\varphi \in H^1(\mathbb{T}_h)$, the broken gradient $\nabla_h : H^1(\mathbb{T}_h) \to L^2(\Omega)^d$ is defined by

$$(\nabla_h \varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h.$$

1.3 Penalty parameters and energy norms

The following trace inequality on anisotropic meshes is significant in this study. Some references can be found for the proof. Here, we follow Ern and Guermond [24, Lemma 12.15].

Lemma 1 (Trace inequality) *Let $T \subset \mathbb{R}^d$ be a simplex. There exists a positive constant c such that for any $\varphi \in H^1(T)$, $F \in \mathcal{F}_T$, and h,*

$$\|\varphi\|_{L^2(F)} \leq c \ell_{T,F}^{-\frac{1}{2}} \left(\|\varphi\|_{L^2(T)} + h_T^{\frac{1}{2}} \|\varphi\|_{L^2(T)}^{\frac{1}{2}} |\varphi|_{H^1(T)}^{\frac{1}{2}} \right), \tag{1.4}$$

where $\ell_{T,F} := \frac{d!|T|_d}{|F|_{d-1}}$ denotes the distance of the vertex of T opposite to F to the face. Furthermore, there exists a positive constant c such that for any $v = (v^{(1)}, \ldots, v^{(d)})^T \in H^1(T)^d$, $F \in \mathcal{F}_T$, and h,

$$\|v\|_{L^2(F)^d} \leq c \ell_{T,F}^{-\frac{1}{2}} \left(\|v\|_{L^2(T)^d} + h_T^{\frac{1}{2}} \|v\|_{L^2(T)^d}^{\frac{1}{2}} |v|_{H^1(T)^d}^{\frac{1}{2}} \right), \tag{1.5}$$

Proof A proof is found in [34, Lemma 1]. □

Deriving an appropriate penalty term is essential in discontinuous Galerkin methods (dG) on anisotropic meshes. The use of weighted averages gives robust dG schemes for various problems; see [21,34,46]. For any $v \in H^1(\mathbb{T}_h)^d$ and $\varphi \in H^1(\mathbb{T}_h)$,

$$[\![((v\varphi) \cdot n]\!]_F = \{\{v\}\}_{\omega,F} \cdot n_F [\![\varphi]\!]_F + [\![v \cdot n]\!]_F \{\{\varphi\}\}_{\overline{\omega},F}. \tag{1.6}$$

For example, if $u \in H_0^1(\Omega) \cap W^{2,1}(\Omega)$, setting $v := -\nabla u$, we have $[\![v \cdot n]\!]_F = 0$ for all $F \in \mathcal{F}_h^i$, see [46, Lemma 4.3]. Using the trace (Lemma 1) and the

Hölder inequalities, the weighted average gives the following estimate for the term $\{\{v\}\}_{\omega,F} \cdot n_F [\![\varphi]\!]_F$.

$$\begin{aligned}
&\int_F |\{\{v\}\}_{\omega,F} \cdot n_F [\![\varphi]\!]_F| \, ds \\
&\leq c h^{\beta} \left(\|v|_{T_1}\|^2_{H^1(T_1)^d} + \|v|_{T_2}\|^2_{H^1(T_2)^d} \right)^{\frac{1}{2}} \\
&\quad \times \left(h^{-2\beta} \omega^2_{T_1,F} \ell_{T_1,F}{}^{-1} + h^{-2\beta} \omega^2_{T_2,F} \ell_{T_2,F}{}^{-1} \right)^{\frac{1}{2}} \|[\![\varphi]\!]\|_{L^2(F)}, \qquad (1.7)
\end{aligned}$$

where $\ell_{T_1,F}$ and $\ell_{T_2,F}$ are defined in the inequality (1.5). The weights $\omega_{T_1,F}$, $\omega_{T_2,F}$ and β are nonnegative real numbers chosen latter on. A choice for the weighted parameters is such that for $F \in \mathcal{F}_h^i$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$,

$$\omega_{T_i,F} := \frac{\sqrt{\ell_{T_i,F}}}{\sqrt{\ell_{T_1,F}} + \sqrt{\ell_{T_2,F}}}, \quad i = 1, 2. \qquad (1.8)$$

Then, the associated penalty parameter is defined as

$$\frac{2h^{-2\beta}}{(\sqrt{\ell_{T_1,F}} + \sqrt{\ell_{T_2,F}})^2}. \qquad (1.9)$$

Let $F \in \mathcal{F}_h^i$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$ be an interior face and $F \in \mathcal{F}_h^{\partial}$ with $F = \partial T_{\partial} \cap \partial \Omega$, $T_{\partial} \in \mathbb{T}_h$ a boundary face. A new penalty parameter κ_F for the WOPSIP method is defined as follows using (1.9) with $\beta = 1$.

$$\kappa_F := \begin{cases} h^{-2} \left(\sqrt{\ell_{T_1,F}} + \sqrt{\ell_{T_2,F}} \right)^{-2} & \text{if } F \in \mathcal{F}_h^i, \\ h^{-2} \ell^{-1}_{T_{\partial},F} \quad \text{if } F \in \mathcal{F}_h^{\partial}. \end{cases} \qquad (1.10)$$

For the RSIP method and the discrete Poincaré inequality, we use the following parameter.

$$\kappa_{F*} := \begin{cases} \left(\sqrt{\ell_{T_1,F}} + \sqrt{\ell_{T_2,F}} \right)^{-2} & \text{if } F \in \mathcal{F}_h^i, \\ \ell^{-1}_{T_{\partial},F} \quad \text{if } F \in \mathcal{F}_h^{\partial}. \end{cases} \qquad (1.11)$$

For any $F \in \mathcal{F}_h$, we define the L^2-projection $\Pi_F^0 : L^2(F) \to \mathbb{P}^0(F)$ by

$$\int_F (\Pi_F^0 \varphi - \varphi) ds = 0 \quad \forall \varphi \in L^2(F).$$

We then define the following norms for any $v \in H^1(\mathbb{T}_h)$.

$$|v|_{wop} := \left(|v|^2_{H^1(\mathbb{T}_h)} + |v|^2_{jwop} \right)^{\frac{1}{2}}$$

with the jump seminorm

$$|v|_{jwop} := \left(\sum_{F \in \mathcal{F}_h} \kappa_F \|\Pi_F^0 [\![v]\!]\|^2_{L^2(F)} \right)^{\frac{1}{2}}$$

and κ_F defined as in (1.10);

$$|v|_{rdg} := \left(|v|^2_{H^1(\mathbb{T}_h)} + |v|^2_{jrdg} \right)^{\frac{1}{2}}$$

with

$$|v|_{jrdg} := \left(\sum_{F \in \mathcal{F}_h} \kappa_{F*} \| \Pi_F^0 [\![v]\!] \|^2_{L^2(F)} \right)^{\frac{1}{2}}$$

and κ_{F*} defined as in (1.11). For any $v \in H^1(\mathbb{T}_h)$, $|v|_{jrdg} \leq |v|_{jwop}$ for $h \leq 1$. The norm $|\cdot|_{wop}$ is used for analysis of the WOPSIP method, while the norm $|\cdot|_{rdg}$ is used for analysis of the RSIP method and the discrete Poincaré inequality (Lemma 6).

Furthermore, for any $k \in \mathbb{N}_0$, let $\mathbb{P}^k(T)$ and $\mathbb{P}^k(F)$ be spaces of polynomials with degree at most k in T and F, respectively.

1.4 Edge characterisation on a simplex, a geometric parameter, and a condition

We impose edge characterisation on a simplex to analyse anisotropic error estimates.

1.4.1 Reference elements

We now define the reference elements $\widehat{T} \subset \mathbb{R}^d$.

Two-dimensional case

Let $\widehat{T} \subset \mathbb{R}^2$ be a reference triangle with vertices $\hat{p}_1 := (0,0)^T$, $\hat{p}_2 := (1,0)^T$, and $\hat{p}_3 := (0,1)^T$.

Three-dimensional case

In the three-dimensional case, we consider the following two cases: (i) and (ii); see Condition 2.

Let $\widehat{T}_1$ and $\widehat{T}_2$ be reference tetrahedra with the following vertices:

(i) $\widehat{T}_1$ has vertices $\hat{p}_1 := (0,0,0)^T$, $\hat{p}_2 := (1,0,0)^T$, $\hat{p}_3 := (0,1,0)^T$, and $\hat{p}_4 := (0,0,1)^T$;
(ii) $\widehat{T}_2$ has vertices $\hat{p}_1 := (0,0,0)^T$, $\hat{p}_2 := (1,0,0)^T$, $\hat{p}_3 := (1,1,0)^T$, and $\hat{p}_4 := (0,0,1)^T$.

Therefore, we set $\widehat{T} \in \{\widehat{T}_1, \widehat{T}_2\}$. Note that the case (i) is called *the regular vertex property*, see [2].

1.4.2 Affine mappings

We introduced a new strategy proposed in [39, Section 2] to use anisotropic mesh partitions. We construct two affine simplex $\widetilde{T} \subset \mathbb{R}^d$, we construct two affine mappings $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ and $\Phi_T : \widetilde{T} \to T$. First, we define the affine mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ as

$$\Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto \tilde{x} := \Phi_{\widetilde{T}}(\hat{x}) := A_{\widetilde{T}} \hat{x} \in \widetilde{T}, \tag{1.12}$$

where $A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$ is an invertible matrix. Section 1.4.3 provides the details. We then define the affine mapping $\Phi_T : \widetilde{T} \to T$ as follows:

$$\Phi_T : \widetilde{T} \ni \tilde{x} \mapsto x := \Phi_T(\tilde{x}) := A_T \tilde{x} + b_T \in T, \tag{1.13}$$

where $b_T \in \mathbb{R}^d$ is a vector and $A_T \in O(d)$ is the rotation and mirror imaging matrix. Section 1.4.4 provides the details. We define the affine mapping $\Phi : \widehat{T} \to T$ as

$$\Phi := \Phi_T \circ \Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto x := \Phi(\hat{x}) = (\Phi_T \circ \Phi_{\widetilde{T}})(\hat{x}) = A\hat{x} + b_T \in T,$$

where $A := A_T A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$.

1.4.3 Construct mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$

We consider affine mapping (1.12). We define the matrix $A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$ as follows: We first define the diagonal matrix as

$$\widehat{A} := \mathrm{diag}(h_1, \ldots, h_d), \quad h_i \in \mathbb{R}_+ \quad \forall i, \tag{1.14}$$

where $\mathbb{R}_+$ denotes the set of positive real numbers.

For $d = 2$, we define the regular matrix $\widetilde{A} \in \mathbb{R}^{2 \times 2}$ as:

$$\widetilde{A} := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \tag{1.15}$$

with parameters

$$s^2 + t^2 = 1, \quad t > 0.$$

For reference element $\widehat{T}$, let $\mathfrak{T}^{(2)}$ be a family of triangles.

$$\widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T}) = A_{\widetilde{T}}(\widehat{T}), \quad A_{\widetilde{T}} := \widetilde{A}\widehat{A}$$

with vertices $\tilde{p}_1 := (0,0)^T$, $\tilde{p}_2 := (h_1, 0)^T$, and $\tilde{p}_3 := (h_2 s, h_2 t)^T$. Then, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$ and $h_2 = |\tilde{p}_1 - \tilde{p}_3| > 0$.

For $d = 3$, we define the regular matrices $\widetilde{A}_1, \widetilde{A}_2 \in \mathbb{R}^{3 \times 3}$ as

$$\widetilde{A}_1 := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \widetilde{A}_2 := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix} \tag{1.16}$$

with parameters

$$\begin{cases} s_1^2 + t_1^2 = 1, \ s_1 > 0, \ t_1 > 0, \ h_2 s_1 \leq h_1/2, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, \ t_2 > 0, \ h_3 s_{21} \leq h_1/2. \end{cases}$$

Therefore, we set $\widetilde{A} \in \{\widetilde{A}_1, \widetilde{A}_2\}$. For the reference elements $\widehat{T}_i$, $i = 1, 2$, let $\mathfrak{T}_i^{(3)}$ and $i = 1, 2$ be a family of tetrahedra.

$$\widetilde{T}_i = \Phi_{\widetilde{T}_i}(\widehat{T}_i) = A_{\widetilde{T}_i}(\widehat{T}_i), \quad A_{\widetilde{T}_i} := \widetilde{A}_i \widehat{A}, \quad i = 1, 2,$$

with vertices

$$\tilde{p}_1 := (0,0,0)^T, \ \tilde{p}_2 := (h_1, 0, 0)^T, \ \tilde{p}_4 := (h_3 s_{21}, h_3 s_{22}, h_3 t_2)^T,$$
$$\begin{cases} \tilde{p}_3 := (h_2 s_1, h_2 t_1, 0)^T & \text{for case (i)}, \\ \tilde{p}_3 := (h_1 - h_2 s_1, h_2 t_1, 0)^T & \text{for case (ii)}. \end{cases}$$

Subsequently, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$, $h_3 = |\tilde{p}_1 - \tilde{p}_4| > 0$, and

$$h_2 = \begin{cases} |\tilde{p}_1 - \tilde{p}_3| > 0 & \text{for case (i)}, \\ |\tilde{p}_2 - \tilde{p}_3| > 0 & \text{for case (ii)}. \end{cases}$$

1.4.4 Construct mapping $\Phi_T : \widetilde{T} \to T$

We determine the affine mapping (1.13) as follows: Let $T \in \mathbb{T}_h$ have vertices p_i $(i = 1, \ldots, d+1)$. Let $b_T \in \mathbb{R}^d$ be the vector and $A_T \in O(d)$ be the rotation and mirror imaging matrix such that

$$p_i = \Phi_T(\tilde{p}_i) = A_T \tilde{p}_i + b_T, \quad i \in \{1, \ldots, d+1\},$$

where the vertices p_i $(i = 1, \ldots, d+1)$ satisfy the following conditions:

Condition 1 (Case in which $d = 2$) *Let $T \in \mathbb{T}_h$ with the vertices p_i $(i = 1, \ldots, 3)$. We assume that $\overline{p_2 p_3}$ is the longest edge of T; i.e., $h_T := |p_2 - p_3|$. We set $h_1 = |p_1 - p_2|$ and $h_2 = |p_1 - p_3|$. We then assume that $h_2 \leq h_1$. Note that $h_1 \approx h_T$.*

Condition 2 (Case in which $d = 3$) *Let $T \in \mathbb{T}_h$ with the vertices p_i $(i = 1, \ldots, 4)$. Let L_i $(1 \leq i \leq 6)$ be the edges of T. We denote by $L_{\min}$ the edge of T with the minimum length; i.e., $|L_{\min}| = \min_{1 \leq i \leq 6} |L_i|$. We set $h_2 := |L_{\min}|$ and assume that*

$$\textit{the endpoints of } L_{\min} \textit{ are either } \{p_1, p_3\} \textit{ or } \{p_2, p_3\}.$$

Among the four edges that share an endpoint with $L_{\min}$, we take the longest edge $L_{\max}^{(\min)}$. Let p_1 and p_2 be the endpoints of edge $L_{\max}^{(\min)}$. We thus have that

$$h_1 = |L_{\max}^{(\min)}| = |p_1 - p_2|.$$

We consider cutting $\mathbb{R}^3$ with the plane that contains the midpoint of edge $L_{\max}^{(\min)}$ and is perpendicular to the vector $p_1 - p_2$. Thus, we have two cases:

(Type i) p_3 *and* p_4 *belong to the same half-space;*
(Type ii) p_3 *and* p_4 *belong to different half-spaces.*

In each case, we set

(Type i) p_1 *and* p_3 *as the endpoints of* $L_{\min}$*, that is,* $h_2 = |p_1 - p_3|$*;*
(Type ii) p_2 *and* p_3 *as the endpoints of* $L_{\min}$*, that is,* $h_2 = |p_2 - p_3|$*.*

Finally, we set $h_3 = |p_1 - p_4|$*. Note that we implicitly assume that* p_1 *and* p_4 *belong to the same half-space. In addition, note that* $h_1 \approx h_T$*.*

1.5 Additional notation and assumption

We define vectors $r_n \in \mathbb{R}^d$, $n = 1, \ldots, d$ as follows. If $d = 2$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_2 := \frac{p_3 - p_1}{|p_3 - p_1|},$$

and if $d = 3$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_3 := \frac{p_4 - p_1}{|p_4 - p_1|}, \quad \begin{cases} r_2 := \dfrac{p_3 - p_1}{|p_3 - p_1|}, & \text{for case (i)}, \\ r_2 := \dfrac{p_3 - p_2}{|p_3 - p_2|} & \text{for case (ii)}. \end{cases}$$

For a sufficiently smooth function φ and vector function $v := (v_1, \ldots, v_d)^T$, we define the directional derivative as, for $i \in \{1, \ldots, d\}$,

$$\frac{\partial \varphi}{\partial r_i} := (r_i \cdot \nabla_x)\varphi = \sum_{i_0=1}^{d} (r_i)_{i_0} \frac{\partial \varphi}{\partial x_{i_0}},$$

$$\frac{\partial v}{\partial r_i} := \left(\frac{\partial v_1}{\partial r_i}, \ldots, \frac{\partial v_d}{\partial r_i} \right)^T = ((r_i \cdot \nabla_x)v_1, \ldots, (r_i \cdot \nabla_x)v_d)^T.$$

For a multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, we use the notation

$$\partial^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}}, \quad \partial_r^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial r_1^{\beta_1} \ldots \partial r_d^{\beta_d}}, \quad h^\beta := h_1^{\beta_1} \cdots h_d^{\beta_d}.$$

Note that $\partial^\beta \varphi \neq \partial_r^\beta \varphi$.

We proposed a geometric parameter H_T in a prior work [36].

Definition 1 *The parameter* H_T *is defined as*

$$H_T := \frac{\prod_{i=1}^d h_i}{|T|_d} h_T.$$

We introduce the geometric condition proposed in [36], which is equivalent to the maximum-angle condition [38].

Assumption 1 *A family of meshes* $\{\mathbb{T}_h\}$ *has a semi-regular property if there exists* $\gamma_0 > 0$ *such that*

$$\frac{H_T}{h_T} \leq \gamma_0 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{1.17}$$

1.6 Piola transformations

The Piola transformation $\Psi : L^1(\widehat{T})^d \to L^1(T)^d$ is defined as

$$\begin{aligned} \Psi : L^1(\widehat{T})^d &\to L^1(T)^d \\ \hat{v} \mapsto v(x) &:= \Psi(\hat{v})(x) = \frac{1}{\det(A)} A\hat{v}(\hat{x}). \end{aligned}$$

1.7 Finite element spaces and anisotropic interpolation error estimates

1.7.1 Finite element spaces

For $s \in \mathbb{N}_0$, we define a discontinuous finite element space as

$$P^s_{dc,h} := \left\{ p_h \in L^2(\Omega);\ p_h|_T \circ \Phi \in \mathbb{P}^s(\widehat{T}) \quad \forall T \in \mathbb{T}_h \right\}.$$

Let Ne be the number of elements included in the mesh $\mathbb{T}_h$. Thus, we write $\mathbb{T}_h = \{T_j\}_{j=1}^{Ne}$.

Let the points $\{P_{T_j,1}, \dots, P_{T_j,d+1}\}$ be the vertices of the simplex $T_j \in \mathbb{T}_h$ for $j \in \{1, \dots, Ne\}$. Let $F_{T_j,i}$ be the face of T_j opposite $P_{T_j,i}$ for $i \in \{1, \dots, d+1\}$. We set $P := \mathbb{P}^1$, and take a set $\Sigma_{T_j} := \{\chi^{CR}_{T_j,i}\}_{1 \le i \le d+1}$ of linear forms with its components such that for any $p \in \mathbb{P}^1$.

$$\chi^{CR}_{T_j,i}(p) := \frac{1}{|F_{T_j,i}|_{d-1}} \int_{F_{T_j,i}} p ds \quad \forall i \in \{1, \dots, d+1\}. \tag{1.18}$$

For each $j \in \{1, \dots, Ne\}$, the triple $\{T_j, \mathbb{P}^1, \Sigma_{T_j}\}$ is a finite element. Using the barycentric coordinates $\{\lambda_{T_j,i}\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ on the reference element, the nodal basis functions associated with the degrees of freedom by (1.18) are defined as

$$\theta^{CR}_{T_j,i}(x) := d\left(\frac{1}{d} - \lambda_{T_j,i}(x)\right) \quad \forall i \in \{1, \dots, d+1\}. \tag{1.19}$$

For $j \in \{1, \dots, Ne\}$ and $i \in \{1, \dots, d+1\}$, we define the function $\phi_{j(i)}$ as

$$\phi_{j(i)}(x) := \begin{cases} \theta^{CR}_{T_j,i}(x), & x \in T_j, \\ 0, & x \notin T_j. \end{cases} \tag{1.20}$$

We define a discontinuous finite element space as

$$V^{CR}_{dc,h} := \left\{ \sum_{j=1}^{Ne} \sum_{i=1}^{d+1} c_{j(i)} \phi_{j(i)};\ c_{j(i)} \in \mathbb{R},\ \forall i, j \right\} \subset P^1_{dc,h} \tag{1.21}$$

with a norm $|\varphi_h|_{V^{CR}_{dc,h}} := |\varphi_h|_h$ for any $\varphi_h \in V^{CR}_{dc,h}$.

For $T_j \in \mathbb{T}_h,\ j \in \{1,\dots,Ne\}$, we define the local RT polynomial space as follows.

$$\mathbb{RT}^0(T_j) := \mathbb{P}^0(T_j)^d + x\mathbb{P}^0(T_j), \quad x \in \mathbb{R}^d. \tag{1.22}$$

For $p \in \mathbb{RT}^0(T_j)$, the local degrees of freedom are defined as

$$\chi^{RT}_{T_j,i}(p) := \int_{F_{T_j,i}} p \cdot n_{T_j,i} ds \quad \forall i \in \{1,\dots,d+1\}, \tag{1.23}$$

where $n_{T_j,i}$ is a fixed unit normal to $F_{T_j,i}$. Setting $\Sigma^{RT}_{T_j} := \{\chi^{RT}_{T_j,i}\}_{1\le i\le d+1}$, the triple $\{T_j, \mathbb{RT}^0, \Sigma^{RT}_{T_j}\}$ a finite element. The local shape functions are

$$\theta^{RT}_{T_j,i}(x) := \frac{\iota_{F_{T_j,i},T_j}}{d|T_j|_d}(x - P_{T_j,i}) \quad \forall i \in \{1,\dots,d+1\}, \tag{1.24}$$

where $\iota_{F_{T_j,i},T_j} := 1$ if $n_{T_j,i}$ points outwards, and -1 otherwise [24]. We define a discontinuous RT finite element space as follows.

$$V^{RT}_{dc,h} := \{v_h \in L^1(\Omega)^d : \ \Psi^{-1}(v_h|_{T_j}) \in \mathbb{RT}^0(\widehat{T}) \quad \forall T_j \in \mathbb{T}_h,\ j \in \{1,\dots,Ne\}\}. \tag{1.25}$$

1.7.2 Discontinuous space and the L^2-orthogonal projection

For $T_j \in \mathbb{T}_h,\ j \in \{1,\dots,Ne\}$, let $\Pi^0_{T_j} : L^2(T_j) \to \mathbb{P}^0$ be the L^2-orthogonal projection defined as

$$\Pi^0_{T_j}\varphi := \frac{1}{|T_j|_d}\int_{T_j} \varphi dx \quad \forall \varphi \in L^2(T_j).$$

The following theorem gives an anisotropic error estimate of the projection $\Pi^0_{T_j}$.

Theorem 1 *For any $\hat{\varphi} \in H^1(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$,*

$$\|\Pi^0_{T_j}\varphi - \varphi\|_{L^2(T_j)} \le c \sum_{i=1}^{d} h_i \left\| \frac{\partial \varphi}{\partial r_i} \right\|_{L^2(T_j)}. \tag{1.26}$$

Proof A proof is found in [34, Theorem 2] and [35, Theorem 2]. □

We also define the global interpolation Π^0_h to the space $P^0_{dc,h}$ as

$$(\Pi^0_h\varphi)|_{T_j} := \Pi^0_{T_j}(\varphi|_{T_j}) \ \forall T_j \in \mathbb{T}_h,\ j \in \{1,\dots,Ne\},\ \forall \varphi \in L^2(\Omega).$$

1.7.3 Discontinuous CR finite element interpolation operator

For $T_j \in \mathbb{T}_h$, $j \in \{1,\dots,Ne\}$, let $I_{T_j}^{CR} : H^1(T_j) \to \mathbb{P}^1(T_j)$ be the CR interpolation operator such that for any $\varphi \in H^1(T_j)$,

$$I_{T_j}^{CR} : H^1(T_j) \ni \varphi \mapsto I_{T_j}^{CR}\varphi := \sum_{i=1}^{d+1} \left(\frac{1}{|F_{T_j,i}|_{d-1}} \int_{F_{T_j,i}} \varphi ds \right) \theta_{T_j,i}^{CR} \in \mathbb{P}^1(T_j).$$

We then present estimates of the anisotropic CR interpolation error.

Theorem 2 *For $j \in \{1,\dots,Ne\}$,*

$$|I_{T_j}^{CR}\varphi - \varphi|_{H^1(T_j)} \le c \sum_{i=1}^{d} h_i \left\| \frac{\partial}{\partial r_i} \nabla \varphi \right\|_{L^2(T_j)^d} \quad \forall \varphi \in H^2(T_j), \tag{1.27}$$

$$\|I_{T_j}^{CR}\varphi - \varphi\|_{L^2(T_j)} \le c \sum_{|\varepsilon|=2} h^{\varepsilon} \|\partial_r^{\varepsilon}\varphi\|_{L^2(T)} \quad \forall \varphi \in H^2(T_j). \tag{1.28}$$

Proof The proof of (1.27) is found in [34, Theorem 3] and [35, Theorem 3].

Let $\varphi \in H^2(T_j)$ for $j \in \{1,\dots,Ne\}$. Using the scaling argument yields

$$\|I_{T_j}^{CR}\varphi - \varphi\|_{L^2(T)} \le c|\det(A)|^{\frac{1}{2}} \|I_{\widehat{T}}^{CR}\hat{\varphi} - \hat{\varphi}\|_{L^2(\widehat{T})}. \tag{1.29}$$

For any $\hat{\eta} \in \mathbb{P}^1$, we have that

$$\|I_{\widehat{T}}^{CR}\hat{\varphi} - \hat{\varphi}\|_{L^2(\widehat{T})} \le \|I_{\widehat{T}}^{CR}(\hat{\varphi} - \hat{\eta})\|_{L^2(\widehat{T})} + \|\hat{\eta} - \hat{\varphi}\|_{L^2(\widehat{T})}, \tag{1.30}$$

because $I_{\widehat{T}}^{CR}\hat{\eta} = \hat{\eta}$. Using the trace inequality on the reference element,

$$\|I_{\widehat{T}}^{CR}(\hat{\varphi} - \hat{\eta})\|_{L^2(\widehat{T})} \le c\|\hat{\varphi} - \hat{\eta}\|_{H^1(\widehat{T})}. \tag{1.31}$$

Based on (1.29), (1.30) and (1.31), we have that

$$\|I_{T_j}^{CR}\varphi - \varphi\|_{L^2(T)} \le c|\det(A)|^{\frac{1}{2}} \inf_{\hat{\eta}\in\mathbb{P}^1} \|\hat{\varphi} - \hat{\eta}\|_{H^1(\widehat{T})}. \tag{1.32}$$

From the Bramble–Hilbert lemma (refer to [18, Lemma 4.3.8]), $\hat{\eta}_\beta \in \mathbb{P}^1$ exists such that for any $\hat{\varphi} \in H^2(\widehat{T})$,

$$|\hat{\varphi} - \hat{\eta}_\beta|_{H^s(\widehat{T})} \le C^{BH}(\widehat{T})|\hat{\varphi}|_{H^2(\widehat{T})}, \quad s = 0, 1. \tag{1.33}$$

Using the inequality in [39, Lemma 6] with $m = 0$, we can estimate inequality (1.33) as

$$|\hat{\varphi}|_{H^2(\widehat{T})} \le c|\det(A)|^{-\frac{1}{2}} \sum_{|\varepsilon|=2} h^{\varepsilon} \|\partial_r^{\varepsilon}\varphi\|_{L^2(T)}. \tag{1.34}$$

Using (1.32), (1.33) and (1.34), we can deduce the target inequality (1.28). □

We define a global interpolation operator $I_h^{CR} : H^1(\Omega) \to V_{dc,h}^{CR}$ as

$$(I_h^{CR}\varphi)|_{T_j} = I_{T_j}^{CR}(\varphi|_{T_j}), \quad j \in \{1,\dots,Ne\}, \quad \forall \varphi \in H^1(\Omega). \tag{1.35}$$

1.7.4 Discontinuous RT finite element interpolation operator

For $T_j \in \mathbb{T}_h$, $j \in \{1, \ldots, Ne\}$, let $\mathcal{I}_{T_j}^{RT} : H^1(T_j)^d \to \mathbb{RT}^0(T_j)$ be the RT interpolation operator such that for any $v \in H^1(T_j)^d$,

$$\mathcal{I}_{T_j}^{RT} : H^1(T_j)^d \ni v \mapsto \mathcal{I}_{T_j}^{RT} v := \sum_{i=1}^{d+1} \left(\int_{F_{T_j,i}} v \cdot n_{T_j,i} ds \right) \theta_{T_j,i}^{RT} \in \mathbb{RT}^0(T_j). \tag{1.36}$$

The following two theorems are divided into the element of (Type i) or the element of (Type ii) in Section 1.4 when $d = 3$.

Theorem 3 *Let T_j be the element with Conditions 1 or 2 and satisfy (Type i) in Section 1.4 when $d = 3$. For any $\hat{v} \in H^1(\widehat{T})^d$ with $v = (v_1, \ldots, v_d)^T := \Psi \hat{v}$ and $j \in \{1, \ldots, Ne\}$,*

$$\|\mathcal{I}_{T_j}^{RT} v - v\|_{L^2(T_j)^d} \leq c \left(\frac{H_{T_j}}{h_{T_j}} \sum_{i=1}^{d} h_i \left\| \frac{\partial v}{\partial r_i} \right\|_{L^2(T_j)^d} + h_{T_j} \| \operatorname{div} v\|_{L^2(T_j)} \right). \tag{1.37}$$

Proof A proof is provided in [33, Theorem 2]. □

Theorem 4 *Let $d = 3$. Let T_j be an element with Condition 2 that satisfies (Type ii) in Section 1.4. For $\hat{v} \in H^1(\widehat{T})^3$ with $v = (v_1, v_2, v_3)^T := \Psi \hat{v}$ and $j \in \{1, \ldots, Ne\}$,*

$$\|\mathcal{I}_{T_j}^{RT} v - v\|_{L^2(T_j)^3} \leq c \frac{H_{T_j}}{h_{T_j}} \left(h_{T_j} |v|_{H^1(T_j)^3} \right). \tag{1.38}$$

Proof A proof is provided in [33, Theorem 3]. □

We define a global interpolation operator $\mathcal{I}_h^{RT} : H^1(\Omega)^d \to V_{dc,h}^{RT}$ by

$$(\mathcal{I}_h^{RT} v)|_{T_j} = \mathcal{I}_{T_j}^{RT} (v|_{T_j}), \quad j \in \{1, \ldots, Ne\}, \quad \forall v \in H^1(\Omega)^d. \tag{1.39}$$

1.8 Relation between the RT interpolation and the L^2-projection

Between the RT interpolation $\mathcal{I}_h^{RT}$ and the L^2-projection Π_h^0, the following relation holds.

Lemma 2 *For $j \in \{1, \ldots, Ne\}$,*

$$\operatorname{div}(\mathcal{I}_{T_j}^{RT} v) = \Pi_{T_j}^0 (\operatorname{div} v) \quad \forall v \in H^1(T_j)^d. \tag{1.40}$$

By combining (1.40), *for any $v \in H^1(\Omega)^d$*

$$\operatorname{div}(\mathcal{I}_h^{RT} v) = \Pi_h^0 (\operatorname{div} v). \tag{1.41}$$

Proof A proof is provided in [24, Lemma 16.2]. □

1.9 Existing results

1.9.1 Important tools

The following relation plays an important role in the discontinuous Galerkin finite element analysis on anisotropic meshes.

Lemma 3 *For any $w \in H^1(\Omega)^d$ and $\psi_h \in P^1_{dc,h}$,*

$$\int_\Omega \left(\mathcal{I}_h^{RT} w \cdot \nabla_h \psi_h + \operatorname{div} \mathcal{I}_h^{RT} w \psi_h \right) dx \\ = \sum_{F \in \mathcal{F}_h^i} \int_F \{\{w\}\}_{\omega,F} \cdot n_F \Pi_F^0 [\![\psi_h]\!]_F ds + \sum_{F \in \mathcal{F}_h^\partial} \int_F (w \cdot n_F) \Pi_F^0 \psi_h ds. \quad (1.42)$$

Proof A proof is provided in [34, Lemma 3]. □

The right-hand terms in (1.42) are estimated as follows.

Lemma 4 *For any $w \in H^1(\Omega)^d$ and $\psi_h \in P^1_{dc,h}$,*

$$\left| \sum_{F \in \mathcal{F}_h^i} \int_F \{\{w\}\}_{\omega,F} \cdot n_F \Pi_F^0 [\![\psi_h]\!]_F ds \right| \\ \leq c |\psi_h|_{jwop} \left(h \|w\|_{L^2(\Omega)^d} + h^{\frac{3}{2}} \|w\|^{\frac{1}{2}}_{L^2(\Omega)^d} |w|^{\frac{1}{2}}_{H^1(\Omega)^d} \right), \quad (1.43)$$

$$\left| \sum_{F \in \mathcal{F}_h^\partial} \int_F (w \cdot n_F) \Pi_F^0 \psi_h ds \right| \\ \leq c |\psi_h|_{jwop} \left(h \|w\|_{L^2(\Omega)^d} + h^{\frac{3}{2}} \|w\|^{\frac{1}{2}}_{L^2(\Omega)^d} |w|^{\frac{1}{2}}_{H^1(\Omega)^d} \right). \quad (1.44)$$

Proof A proof is provided in [34, Lemma 4]. □

Lemma 5 *Let $h \leq 1$. Thus, for any $w \in H^1(\Omega)^d$ and $\psi_h \in P^1_{dc,h}$,*

$$\left| \sum_{F \in \mathcal{F}_h^i} \int_F \{\{w\}\}_{\omega,F} \cdot n_F \Pi_F^0 [\![\psi_h]\!]_F ds \right| \leq c |\psi_h|_{rdg} \|w\|_{H^1(\Omega)^d}, \quad (1.45)$$

$$\left| \sum_{F \in \mathcal{F}_h^\partial} \int_F (w \cdot n_F) \Pi_F^0 \psi_h ds \right| \leq c |\psi_h|_{rdg} \|w\|_{H^1(\Omega)^d}. \quad (1.46)$$

Proof A proof is provided in [34, Lemma 4]. □

1.9.2 Discrete Poincaré inequality

The following lemma provides a discrete Poincaré inequality. For simplicity, we assume that Ω is convex.

Lemma 6 (Discrete Poincaré inequality) *Assume that Ω is convex. Let $\{\mathbb{T}_h\}$ be a family of meshes with the semi-regular property (Assumption 1) and $h \leq 1$. Then, there exists a positive constant C_{dc}^P independent of h but dependent on the maximum angle such that*

$$\|\psi_h\|_{L^2(\Omega)} \leq C_{dc}^P |\psi_h|_{rdg} \quad \forall \psi_h \in P_{dc,h}^1. \tag{1.47}$$

Proof A proof is provided in [34, Lemma 6]. □

Remark 1 For any $f \in L^2(\Omega)$, we set $\ell_h : P_{dc,h}^1 \to \mathbb{R}$ such that

$$\ell_h(\psi_h) := \int_\Omega f \psi_h dx \quad \forall \psi_h \in P_{dc,h}^1. \tag{1.48}$$

The Hölder's and the discrete Poincaré inequalities yield

$$|\ell_h(\psi_h)| \leq C_{dc}^P \|f\|_{L^2(\Omega)} |\psi_h|_{rdg} \quad \forall \psi_h \in P_{dc,h}^1,$$

if Ω is convex. We are interested in case that Ω is not convex to prove stability estimates of schemes. In [14], the discrete Poincaré inequalities for piecewise H^1 functions are proposed. However, the inverse, trace inequalities and the local quasi-uniformity for meshes under the shape-regular condition are used for the proof. Therefore, careful consideration of the results used in [14] may be necessary to remove the assumption that Ω is convex.

2 WOPSIP method for the Poisson equation

This section presents an analysis of the WOPSIP method for the Poisson equations on anisotropic meshes. In [35], we presented the error estimate in the energy norm $|\cdot|_{wop}$ in Section 1.3 for the Stokes equation. We review the energy norm eror estimate and here is a new introduction to an error estimate in the L^2 norm.

2.1 WOPSIP method

We consider the WOPSIP method for the Poisson equation (1.1) as follows. We aim to find $u_h^{wop} \in V_{dc,h}^{CR}$ such that

$$a_h^{wop}(u_h^{wop}, \varphi_h) = \ell_h(\varphi_h) \quad \forall \varphi_h \in V_{dc,h}^{CR}, \tag{2.1}$$

where $\ell_h : V_{dc,h}^{CR} \to \mathbb{R}$ is defined in (1.48). Here, $a_h^{wop} : (V_{dc,h}^{CR} + H_0^1(\Omega)) \times (V_{dc,h}^{CR} + H_0^1(\Omega))$ is defined as

$$a_h^{wop}(v,w) := \int_\Omega \nabla_h v \cdot \nabla_h w dx + \sum_{F \in \mathcal{F}_h} \kappa_F \int_F \Pi_F^0 [\![v]\!] \Pi_F^0 [\![w]\!] ds$$

for all $(v,w) \in (V_{dc,h}^{CR} + H_0^1(\Omega)) \times (V_{dc,h}^{CR} + H_0^1(\Omega))$. Recall that the parameter κ_F is defined in (1.10). Using the Hölder's inequality, we obtain

$$|a_h^{wop}(v, w_h)| \leq c|v|_{wop}|w_h|_{wop} \quad \forall v \in V_{dc,h}^{CR} + H_0^1(\Omega), \ \forall w_h \in V_{dc,h}^{CR}. \tag{2.2}$$

As stated in Remark 1, we impose that Ω is convex to obtain a stability estimate of the WOPSIP sheme. The stability estimate without convexity of the domain is still an open question.

2.2 Energy norm error estimate

The starting point for error analysis is the Second Strang Lemma, e.g. see [23, Lemma 2.25].

Lemma 7 *We assume that Ω is convex. Let $u \in H_0^1(\Omega)$ be the solution of* (1.2) *and $u_h^{wop} \in V_{dc,h}^{CR}$ be the solution of* (2.1). *It then holds that*

$$|u - u_h^{wop}|_{wop} \leq \inf_{v_h \in V_{dc,h}^{CR}} |u - v_h|_{wop} + E_h(u), \tag{2.3}$$

where

$$E_h(u) := \sup_{w_h \in V_{dc,h}^{CR}} \frac{|a_h^{wop}(u, w_h) - \ell_h(w_h)|}{|w_h|_{wop}}. \tag{2.4}$$

Proof Let $v_h \in V_{dc,h}^{CR}$. It holds that

$$\begin{aligned}
|v_h - u_h^{wop}|_{wop}^2 &= a_h^{wop}(v_h - u_h^{wop}, v_h - u_h^{wop}) \\
&= a_h^{wop}(v_h - u, v_h - u_h^{wop}) + a_h^{wop}(u, v_h - u_h^{wop}) - \ell_h(v_h - u_h^{wop}) \\
&\leq c|u - v_h|_{wop}|v_h - u_h^{wop}|_{wop} + |a_h^{wop}(u, v_h - u_h^{wop}) - \ell_h(v_h - u_h^{wop})|,
\end{aligned}$$

which leads to

$$\begin{aligned}
|v_h - u_h^{wop}|_{wop} &\leq c|u - v_h|_{wop} + \frac{|a_h^{wop}(u, v_h - u_h^{wop}) - \ell_h(v_h - u_h^{wop})|}{|v_h - u_h^{wop}|_{wop}} \\
&\leq c|u - v_h|_{wop} + E_h(u).
\end{aligned}$$

Then,

$$|u - u_h^{wop}|_{wop} \leq |u - v_h|_{wop} + |v_h - u_h^{wop}|_{wop} \leq c|u - v_h|_{wop} + E_h(u).$$

Hence, the target inequality (2.3) holds. □

Lemma 8 (Best approximation) *We assume that Ω is convex. Let $u \in V_* := H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of (1.2). Then,*

$$\inf_{v_h \in V_{dc,h}^{CR}} |u - v_h|_{wop} \leq c \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}}. \tag{2.5}$$

Proof Let $v \in H_0^1(\Omega)$. From [35, Theorem 7],

$$\| \Pi_F^0 [\![(I_h^{CR} v) - v]\!] \|_{L^2(F)}^2 = 0. \tag{2.6}$$

Therefore, using (1.27) and (1.44), we obtain

$$\begin{aligned} \inf_{v_h \in V_{dc,h}^{CR}} |u - v_h|_{wop} &\leq |u - I_h^{CR} u|_{wop} = |u - I_h^{CR} u|_{H^1(\mathbb{T}_h)} \\ &\leq c \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{2.7}$$

which is the target inequality (2.5). □

The essential part for error estimates is the consistency error term (2.4).

Lemma 9 (Asymptotic Consistency) *We assume that Ω is convex. Let $u \in V_* := H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of (1.2). Let $\{\mathbb{T}_h\}$ be a family of conformal meshes with the semi-regular property (Assumption 1). Let $T \in \mathbb{T}_h$ be the element with Conditions 1 or 2 and satisfy (Type i) in Section 1.4 when $d = 3$. Then,*

$$\begin{aligned} E_h(u) &\leq c \left\{ \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h \|\Delta u\|_{L^2(\Omega)} \right\} \\ &\quad + c \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right). \end{aligned} \tag{2.8}$$

Furthermore, let $d = 3$ and let $T \in \mathbb{T}_h$ be the element with Condition 2 and satisfy (Type ii) in Section 1.4. It then holds that

$$E_h(u) \leq ch \|\Delta u\|_{L^2(\Omega)} + c \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right). \tag{2.9}$$

Proof We first have

$$\operatorname{div} \mathcal{I}_h^{RT} \nabla u = \Pi_h^0 \operatorname{div} \nabla u = \Pi_h^0 \Delta u.$$

Let $w_h \in V_{dc,h}^{CR}$. Setting $w := \nabla u$ in (1.42) yields

$$\begin{aligned} & a_h^{wop}(u, w_h) - \ell_h(w_h) \\ &= \int_\Omega (\nabla u - \mathcal{I}_h^{RT} \nabla u) \cdot \nabla_h w_h dx + \int_\Omega \left(\Delta u - \Pi_h^0 \Delta u \right) w_h dx \\ &\quad + \sum_{F \in \mathcal{F}_h^i} \int_F \{\{ \nabla u \}\}_{\omega, F} \cdot n_F \Pi_F^0 [\![w_h]\!]_F ds + \sum_{F \in \mathcal{F}_h^\partial} \int_F (\nabla u \cdot n_F) \Pi_F^0 w_h ds \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let $T \in \mathbb{T}_h$ be the element with Conditions 1 or 2 and satisfy (Type i) in Section 1.4 when $d = 3$. Using the Hölder's inequality, the Cauchy–Schwarz inequality and the RT interpolation error (1.37), the term I_1 is estimated as

$$\begin{aligned}
|I_1| &\leq c \sum_{T \in \mathbb{T}_h} \|\nabla u - \mathcal{I}_h^{RT} \nabla u\|_{L^2(T)} |w_h|_{H^1(T)} \\
&\leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{i=1}^{d} h_i \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d} + h_T \|\Delta u\|_{L^2(T)} \right) |w_h|_{H^1(T)} \\
&\leq c \left\{ \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h \|\Delta u\|_{L^2(\Omega)} \right\} |w_h|_{wop}.
\end{aligned}$$

Using the Hölder's inequality, the Cauchy–Schwarz inequality, the stability of Π_h^0 and the estimate (1.26), the term I_2 is estimated as

$$\begin{aligned}
|I_2| &= \left| \int_{\Omega} \left(\Delta u - \Pi_h^0 \Delta u \right) \left(w_h - \Pi_h^0 w_h \right) dx \right| \\
&\leq \sum_{T \in \mathbb{T}_h} \|\Delta u - \Pi_h^0 \Delta u\|_{L^2(T)} \|w_h - \Pi_h^0 w_h\|_{L^2(T)} \\
&\leq ch \|\Delta u\|_{L^2(\Omega)} |w_h|_{wop}.
\end{aligned}$$

The triangle inequality, (1.43) and (1.44), the terms I_3 and I_4 are estimated as

$$\begin{aligned}
|I_3| &\leq c |w_h|_{jwop} \left(h \|\nabla u\|_{L^2(\Omega)^d} + h^{\frac{3}{2}} \|\nabla u\|_{L^2(\Omega)^d}^{\frac{1}{2}} |\nabla u|_{H^1(\Omega)^d}^{\frac{1}{2}} \right) \\
&\leq c |w_h|_{wop} \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right), \\
|I_4| &\leq c |w_h|_{wop} \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right).
\end{aligned}$$

Gathering the above inequalities yields the target inequality (2.8).

Let $d = 3$ and let $T \in \mathbb{T}_h$ be the element with Condition 2 and satisfy (Type ii) in Section 1.4. Using the Hölder's inequality, the Cauchy–Schwarz inequality and the RT interpolation error (1.38), the terms I_1 is estimated as

$$|I_1| \leq ch |u|_{H^2(\Omega)} |w_h|_{wop},$$

which implies the target inequality (2.9) with (1.3). □

From Lemmata 7, 8 and 9, we have the following energy norm error estimate.

Theorem 5 *We assume that Ω is convex. Let $u \in V_*$ be the solution of* (1.2)*. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes with the semi-regular property (Assumption 1). Let $T \in \mathbb{T}_h$ be the element with Conditions 1 or 2 and satisfy (Type i) in Section 1.4 when $d = 3$. Let $u_h^{wop} \in V_{dc,h}^{CR}$ be the solution of* (2.1)*. Then,*

$$|u - u_h^{wop}|_{wop} \leq c \left\{ \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h \|\Delta u\|_{L^2(\Omega)} \right\} + c \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right). \quad (2.10)$$

Furthermore, let $d = 3$ and let $T \in \mathbb{T}_h$ be the element with Condition 2 and satisfy (Type ii) in Section 1.4. It then holds that

$$|u - u_h^{wop}|_{wop} \leq ch \|\Delta u\|_{L^2(\Omega)} + c \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right). \quad (2.11)$$

2.3 L^2 norm error estimate

This section presents the L^2 error estimate of the WOPSIP method.

Theorem 6 *We assume that Ω is convex. Let $u \in V_*$ be the solution of* (1.2)*. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes with the semi-regular property (Assumption 1). Let $T \in \mathbb{T}_h$ be the element with Conditions 1 or 2 and satisfy (Type i) in Section 1.4 when $d = 3$. Let $u_h^{wop} \in V_{dc,h}^{CR}$ be the solution of* (2.1)*. Then,*

$$\|u - u_h^{wop}\|_{L^2(\Omega)} \leq ch \left\{ \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h \|\Delta u\|_{L^2(\Omega)} \right\} + ch \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right). \quad (2.12)$$

Furthermore, let $d = 3$ and let $T \in \mathbb{T}_h$ be the element with Condition 2 and satisfy (Type ii) in Section 1.4. It then holds that

$$\|u - u_h^{wop}\|_{L^2(\Omega)} \leq ch^2 \|\Delta u\|_{L^2(\Omega)} + ch \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right). \quad (2.13)$$

Proof We set $e := u - u_h^{wop}$. Let $z \in V_*$ satisfy

$$a(\varphi, z) = \int_\Omega \varphi e dx \quad \forall \varphi \in H_0^1(\Omega), \quad (2.14)$$

and $z_h^{wop} \in V_{dc,h}^{CR}$ satisfy

$$a_h^{wop}(\varphi_h, z_h^{wop}) = \int_\Omega \varphi_h e dx \quad \forall \varphi_h \in V_{dc,h}^{CR}. \tag{2.15}$$

Note that $|z|_{H^1(\Omega)} \leq c\|e\|_{L^2(\Omega)}$ and $|z|_{H^2(\Omega)} \leq \|e\|_{L^2(\Omega)}$. Then,

$$\begin{aligned}
\|e\|_{L^2(\Omega)}^2 &= \int_\Omega (u - u_h^{wop}) e dx = a(u,z) - a_h^{wop}(u_h^{wop}, z_h^{wop}) \\
&= a_h^{wop}(u - u_h^{wop}, z - z_h^{wop}) + a_h^{wop}(u - u_h^{wop}, z_h^{wop}) + a_h^{wop}(u_h^{wop}, z - z_h^{wop}) \\
&= a_h^{wop}(u - u_h^{wop}, z - z_h^{wop}) \\
&\quad + a_h^{wop}(u - u_h^{wop}, z_h^{wop} - I_h^{CR} z) + a_h^{wop}(u - u_h^{wop}, I_h^{CR} z) \\
&\quad + a_h^{wop}(u_h^{wop} - I_h^{CR} u, z - z_h^{wop}) + a_h^{wop}(I_h^{CR} u, z - z_h^{wop}) \\
&=: J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

From Theorems 2 and 5, we have

$$|z - z_h^{wop}|_{wop} \leq ch\|e\|_{L^2(\Omega)}, \tag{2.16a}$$

$$|z - I_h^{CR} z|_{H^1(\mathbb{T}_h)} \leq ch\|e\|_{L^2(\Omega)}, \tag{2.16b}$$

$$\|z - I_h^{CR} z\|_{L^2(\Omega)} \leq ch^2\|e\|_{L^2(\Omega)}. \tag{2.16c}$$

Let $T \in \mathbb{T}_h$ be the element with Conditions 1 or 2 and satisfy (Type i) in Section 1.4 when $d = 3$.

Using (2.10) and (2.16a), J_1 can be estimated as

$$\begin{aligned}
|J_1| &\leq c|u - u_h^{wop}|_{wop} |z - z_h^{wop}|_{wop} \\
&\leq ch\|e\|_{L^2(\Omega)} \left\{ \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h\|\Delta u\|_{L^2(\Omega)} \right\} \\
&\quad + ch\|e\|_{L^2(\Omega)} \left(h|u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right).
\end{aligned}$$

Using (2.7), (2.10), (2.16a) and (2.16b), J_2 can be estimated as

$$\begin{aligned}
|J_2| &\leq c \left\{ |u - u_h^{wop}|_{wop} \left(|z_h^{wop} - z|_{wop} + |z - I_h^{CR} z|_{wop} \right) \right\} \\
&\leq ch\|e\|_{L^2(\Omega)} \left\{ \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h\|\Delta u\|_{L^2(\Omega)} \right\} \\
&\quad + ch\|e\|_{L^2(\Omega)} \left(h|u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right).
\end{aligned}$$

By an analogous argument,

$$\begin{aligned}
|J_4| &\leq ch\|e\|_{L^2(\Omega)} \left\{ \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h\|\Delta u\|_{L^2(\Omega)} \right\} \\
&\quad + ch\|e\|_{L^2(\Omega)} \left(h|u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right).
\end{aligned}$$

Using the same argument with (1.42) with $w := \nabla u$ and $\Pi_F^0 [\![(I_h^{CR} z) - z]\!] = 0$,

$$\int_\Omega \left(\mathcal{I}_h^{RT} \nabla u \cdot \nabla_h (I_h^{CR} z - z) + \Pi_h^0 \Delta u (I_h^{CR} z - z) \right) dx = 0.$$

Thus,

$$\begin{aligned} J_3 &= a_h^{wop}(u, I_h^{CR} z) - a_h^{wop}(u_h^{wop}, I_h^{CR} z) \\ &= \int_\Omega \nabla u \cdot \nabla_h (I_h^{CR} z - z) dx + \int_\Omega \Delta u (I_h^{CR} z - z) dx \\ &= \int_\Omega (\nabla u - \mathcal{I}_h^{RT} \nabla u) \cdot \nabla_h (I_h^{CR} z - z) dx + \int_\Omega (\Delta u - \Pi_h^0 \Delta u)(I_h^{CR} z - z) dx. \end{aligned}$$

Using (1.37), (2.16b), (2.16c) and the stanility of the L^2-projection yields

$$\begin{aligned} |J_3| &\le ch \|e\|_{L^2(\Omega)} \left\{ \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + h \|\Delta u\|_{L^2(\Omega)} \right\} \\ &\quad + ch^2 \|e\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)}. \end{aligned}$$

By an analogous argument,

$$\begin{aligned} |J_5| &= \left| \int_\Omega (\nabla z - \mathcal{I}_h^{RT} \nabla z) \cdot \nabla_h (I_h^{CR} u - u) dx + \int_\Omega (\Delta z - \Pi_h^0 \Delta z)(I_h^{CR} u - u) dx \right| \\ &\le ch \|e\|_{L^2(\Omega)} \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} h_i^2 \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)^d}^2 \right)^{\frac{1}{2}} + ch^2 \|e\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)}. \end{aligned}$$

Gathering the above inequalities yields the target inequality (2.12).

Let $d = 3$ and let $T \in \mathbb{T}_h$ be the element with Condition 2 and satisfy (Type ii) in Section 1.4.

We use (2.11) instead of (2.10), and (1.38) instead of (1.37) for estimates of $J_1 - J_5$:

$$\begin{aligned} |J_1| &\le ch^2 \|e\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)} + ch \|e\|_{L^2(\Omega)} \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right), \\ |J_2| &\le ch^2 \|e\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)} + ch \|e\|_{L^2(\Omega)} \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right), \\ |J_3| &\le ch^2 \|e\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)}, \\ |J_4| &\le ch^2 \|e\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)} + ch \|e\|_{L^2(\Omega)} \left(h |u|_{H^1(\Omega)} + h^{\frac{3}{2}} |u|_{H^1(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^2(\Omega)}^{\frac{1}{2}} \right), \\ |J_5| &\le ch^2 \|e\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)}. \end{aligned}$$

Gathering the above inequalities yields the target inequality (2.13). □

3 Remarks on SIP methods

This section gives remaks on SIP and RSIP methods for the Poisson equations without imposing the shape-regular condition for mesh partitions. We define the calassical discontinuous $\mathbb{P}^1$ finite element space as

$$\begin{aligned} V_{dc,h}^1 &:= \{\varphi_h \in L^2(\Omega) : \ v_h|_T \in \mathbb{P}^1(T) \ \forall T \in \mathbb{T}_h\}, \\ V_{dc,h,*}^1 &:= V_* + V_{dc,h}^1, \end{aligned}$$

The SIP and RSIP methods is to find $u_h^\natural \in V_{dc,h}^1$ such that

$$a_h^\natural(u_h^\natural, \varphi_h) = \ell_h(\varphi_h) \quad \forall \varphi_h \in V_{dc,h}^1, \tag{3.1}$$

where $\ell_h : P_{dc,h}^1 \to \mathbb{R}$ is defined in (1.48) and $\natural \in \{sip, rsip\}$. Here, $a_h^{sip} : V_{dc,h,*}^1 \times V_{dc,h}^1$ and $a_h^{rsip} : V_{dc,h,*}^1 \times V_{dc,h}^1$ are defined as

$$\begin{aligned} a_h^{sip}(v, w_h) &:= \int_\Omega \nabla_h v \cdot \nabla_h w_h dx + \sum_{F \in \mathcal{F}_h} \gamma^{sip} \kappa_{F*} \int_F [\![v]\!][\![w_h]\!] ds \\ &\quad - \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{\nabla_h v\}\!\}_\omega \cdot n_F [\![w_h]\!] + [\![v]\!] \{\!\{\nabla_h w_h\}\!\}_\omega \cdot n_F \right) ds, \\ a_h^{rsip}(v, w_h) &:= \int_\Omega \nabla_h v \cdot \nabla_h w_h dx + \sum_{F \in \mathcal{F}_h} \gamma^{rsip} \kappa_{F*} \int_F \Pi_F^0 [\![v]\!] \Pi_F^0 [\![w_h]\!] ds \\ &\quad - \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{\nabla_h v\}\!\}_\omega \cdot n_F [\![w_h]\!] + [\![v]\!] \{\!\{\nabla_h w_h\}\!\}_\omega \cdot n_F \right) ds \end{aligned}$$

for all $(v, w_h) \in V_{dc,h,*}^1 \times V_{dc,h}^1$. The penalty parameters γ^{sip} and γ^{rsip} are large enough. One generally estabilish consistency, boundedness for bilinear forms, and discrete coercivity to obtain a convergence analysis for the SIP and RSIP methods, e.g. see [46, Chapter 4]. However, we may not get an optimal order of error estimates for the consistenct term,

$$\sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\}_\omega \cdot n_F [\![w_h]\!] ds \quad \forall (v, w_h) \in V_{dc,h,*}^1 \times V_{dc,h}^1.$$

Suppose that $F \in \mathcal{F}_h^i$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$. Using the trace (Lemma 1) and the Hölder inequalities, the weighted average gives the following estimate for the term $\int_F \{\{\nabla_h v\}\}_\omega \cdot n_F [\![w_h]\!] ds$.

$$
\begin{aligned}
&\left| \int_F \{\{\nabla_h v\}\}_\omega \cdot n_F [\![w_h]\!] ds \right| \\
&\leq c \Bigg\{ \left(\|\nabla v|_{T_1}\|_{L^2(T_1)^d} + h_{T_1}^{\frac{1}{2}} \|\nabla v|_{T_1}\|_{L^2(T_1)^d}^{\frac{1}{2}} |\nabla v|_{T_1}|_{H^1(T_1)^d}^{\frac{1}{2}} \right)^2 \\
&\quad + \left(\|\nabla v|_{T_2}\|_{L^2(T_2)^d} + h_{T_2}^{\frac{1}{2}} \|\nabla v|_{T_2}\|_{L^2(T_2)^d}^{\frac{1}{2}} |\nabla v|_{T_2}|_{H^1(T_2)^d}^{\frac{1}{2}} \right)^2 \Bigg\}^{\frac{1}{2}} \\
&\quad \times \left(\omega_{T_1,F}^2 \ell_{T_1,F}{}^{-1} + \omega_{T_2,F}^2 \ell_{T_2,F}{}^{-1} \right)^{\frac{1}{2}} \|[\![w_h]\!]\|_{L^2(F)},
\end{aligned}
\tag{3.2}
$$

where $\omega_{T_1,F}$ and $\omega_{T_2,F}$ are defined in (1.8), and $\ell_{T_1,F}$ and $\ell_{T_2,F}$ are defined in the inequality (1.5). Let $I_T^L : \mathcal{C}(T) \to \mathbb{P}^1(T)$ for any $T \in \mathbb{T}_h$ be the usual Lagrange interpolation opertor, and let $u \in V_*$ the exact solution of (1.1). We set $v := u - I_T^L u$. When $d = 3$, due to [39, Corolllary 1], ubder the semi-regular condition (1.17),

$$
|u - I_T^L u|_{H^1(T)} \leq c \sum_{i=1}^{3} h_i \left| \frac{\partial u}{\partial r_i} \right|_{H^1(T)},
$$

which leads to

$$
\begin{aligned}
&\|\nabla(u - I_T^L u)\|_{L^2(T)^3} + h_T^{\frac{1}{2}} \|\nabla(u - I_T^L u)\|_{L^2(T)^3}^{\frac{1}{2}} |\nabla(u - I_T^L u)|_{H^1(T)^3}^{\frac{1}{2}} \\
&\leq c \sum_{i=1}^{3} h_i \left| \frac{\partial u}{\partial r_i} \right|_{H^1(T)} + c h_T^{\frac{1}{2}} \left(\sum_{i=1}^{3} h_i \left| \frac{\partial u}{\partial r_i} \right|_{H^1(T)} \right)^{\frac{1}{2}} |u|_{H^2(T)}^{\frac{1}{2}} \\
&\leq c h_T^{\frac{1}{2}} \left(\sum_{i=1}^{3} h_i \left| \frac{\partial u}{\partial r_i} \right|_{H^1(T)} \right)^{\frac{1}{2}} |u|_{H^2(T)}^{\frac{1}{2}} \leq c h_T |u|_{H^2(T)}.
\end{aligned}
$$

Therefore, even if anisotropic meshes are used, the computational efficiency may not increase much for the SIP method.

In case the RSIP method, the estimate on consistency term is more complicated. For all $(v, w_h) \in V_{dc,h,*}^1 \times V_{dc,h}^1$,

$$
\begin{aligned}
&\int_F \{\{\nabla_h v\}\}_\omega \cdot n_F [\![w_h]\!] ds \\
&= \int_F \{\{\nabla_h v\}\}_\omega \cdot n_F \Pi_F^0 [\![w_h]\!] ds + \int_F \{\{\nabla_h v\}\}_\omega \cdot n_F \left([\![w_h]\!] - \Pi_F^0 [\![w_h]\!] \right) ds.
\end{aligned}
\tag{3.3}
$$

The first term of the R.H.S in (3.3) can be estimated as (3.2), where the CR interpolation operator I_T^{CR} is used instead of I_T^L. Therefore, even if anisotropic meshes are used, the computational efficiency for the RSIP method also may not increase much. In an estimate of the secoond term of the R.H.S in (3.3), we use scaling argument on anisotropic meshes, c.f. [37, Lemma 4].

4 Numerical experiments

Under construction.

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